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The mesonic branch of the deformed conifold

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Abstract: Using coordinates that manifest the S^2-S^3 split of the base, we study D3-branes localized on the three-sphere in the Klebanov-Strassler background. We find a numerical solution for the warp factor and show the emergence of the AdS throat near the stack. In the dual gauge theory, this corresponds to an RG flow along the mesonic branch. We demonstrate how the cubic superpotential of the $\mathcal{N} = 4$ SYM theory emerges at the end of the RG flow.

Keywords: D-branes, Gauge-gravity correspondence, Supersymmetric gauge theory.

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Contents

1. Introduction

The AdS/CFT correspondence [1] enunciates that the low energy effective 4d physics on a heavy stack of $D3$-branes at a smooth point in flat space-time is dual to the near horizon limit of the 10d curved (by the D-branes) geometry. More specifically, the original AdS/CFT conjecture proposed a duality between $\mathcal{N} = 4$ SU(N) SYM gauge theory and type IIB supergravity on $AdS_5 \times S^5$.

One powerful way of constructing gauge theories with less supersymmetry is to consider instead stacks of $D3$-branes at the singular tip of a Calabi-Yau cone X_6 [2, 3]. Exactly like in the flat space case the radial coordinate of X_6 is absorbed in the AdS_5 part of the metric and the near horizon geometry becomes $AdS_5 \times Y^5$, where Y^5 is the $5d$ base of X_6. The number of supercharges in the dual gauge theory is completely encoded in Y^5 and is maximal only for $Y^5 = S^5$.

The most notorious and well studied $\mathcal{N} = 1$ example of this kind is the Klebanov-Witten model, which arises from $D3$-branes placed at the tip of the singular conifold. It was argued in [2] that the low energy effective gauge theory living on the stack has a non-trivial RG fixed point and, therefore, is conformal. The gauge group is SU(N) $\times$ SU(N) and the field content consists of four chiral bi-fundamentals $A_{1,2}$ and $B_{1,2}$ that transform in the $(\mathbf{N}, \bar{\mathbf{N}})$ and the $(\mathbf{N}, \bar{\mathbf{N}})$ representations respectively. The theory has also a marginal superpotential $W \propto \mathrm{Tr}\det_{ij} A_i B_j$. The $5d$ base of the conifold $T^{1,1}$ is topologically $S^3 \times S^2$ and has an $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_R$ isometry, which appears also as the global symmetry of the gauge theory. The two SU(2) factors act on A_i's and B_i's respectively and the non-trivial R-symmetry charges are $\frac{1}{2}$ for all the fields. It is straightforward to check that with this assignment the gauge couplings do not run. The theory enjoys also an additional non-geometric baryonic symmetry $\mathrm{U}(1)_B$ and a chiral $\mathbb{Z}_2$ symmetry, which interchanges A_i's and B_i's and also the two SU(2) groups.

The conformal properties of the gauge theory are encoded in the AdS_5 part of the metric. This is evident from the fact that the $4d$ conformal group is isomorphic to the AdS_5 isometry group SO(4, 2). As we have already mentioned, the AdS_5 factor owes its appearance to the conic structure of the $6d$ CY space X_6. It follows therefore, that in order to build a non-conformal extension of the AdS/CFT duality (the so-called non-AdS/non-CFT correspondence [4, 5]) we have to change the conic structure of X_6, while still possibly keeping some of the supersymmetries. For the conifold there are two ways to achieve this goal. The *deformation* changes the complex structure of the conifold, but still keeps the Kähler structure, while the *resolution* of the conifold breaks the Kähler but preserves the complex structure. Though both the deformation and the resolution make the conifold completely regular and smooth, they look different at the tip. In the former case the S^3 of $T^{1,1}$ approaches a finite size and the S^2 shrinks to zero, while in the latter case the situation is exactly the opposite.

The supersymmetric supergravity solution based on the deformed conifold was constructed by Klebanov and Strassler [6] and has since been a subject of intensive research. The solution necessarily incorporates M fractional $D3$-branes, which are actually regular $D5$-branes wrapped on the two-sphere. On the gauge theory side it means that the gauge group is now $SU\left((k+1)M\right) \times \mathrm{SU}(kM)$. The theory, as expected, is not conformal. When one gauge group becomes weakly coupled, the other becomes strongly coupled. Under Seiberg duality, however, the rôles of the couplings are exchanged, while the gauge group becomes $\mathrm{SU}(kM) \times SU\left((k-1)M\right)$. The theory exhibits, therefore, a *cascade* of Seiberg dualities. At each step of the cascade we have $k \to k-1$ and at the last step we arrive at the SU(M) $\mathcal{N}=1$ SYM. It was first suggested by Aharony [7] that the theory is at a specific $\mathbb{Z}_2$-invariant point on the *baryonic* branch $|A| = |B|$. The broken baryonic symmetry $\mathrm{U}(1)_B$ thus implies that the gauge theory has a pseudoscalar Goldstone boson and its massless scalar superpartner. The supergravity dual of these modes was later found in [8]. The baryonic branch allows also for solutions that break the $\mathbb{Z}_2$ symmetry [10]. The corresponding supergravity duals based on the so-called resolved warped deformed conifold were constructed in [9] (see also [10, 11]).

In this paper we want to construct a gravity dual of the *mesonic* branch of the gauge theory. In this case the gauge group is $\mathrm{SU}(\widetilde{N}+M) \times \mathrm{SU}(\widetilde{N})$ and the cascade step is simply given by $\widetilde{N} \to \widetilde{N} - M$. When $\widetilde{N}$ becomes smaller than M no Seiberg dual description exists anymore. Instead the superpotential receives a non-perturbative Affleck-Dine-Seiberg (ADS) contribution and the quantum moduli space describes M copies of the deformed conifold. The $\mathrm{SU}(\widetilde{N}+M)$ gauge group is broken by the meson VEVs, while the deformation parameter of the conifold depends on the strong coupling scale of the surviving SU(M) gauge group. The branch essentially describes $\widetilde{N}$ $D3$-branes moving on the deformed conifold. Actually, a similar branch exists also for $\widetilde{N} > M$. It corresponds to mesons acquiring large enough VEVs, such that the quantum corrections can be captured

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by an ADS like term in the superpotential. We will now have N $D3$-branes moving on the deformed conifold, where N is the value of $\widetilde{N}$ at the specific step of the cascade.

On the supergravity side the setup should include both M fractional branes of the original KS background and N physical $D3$-branes. To produce a regular $10d$ solution we have to localize the $D3$-branes at a point on the conifold. We will be interested in a $D3$-brane stack placed at the "tip", namely the North pole of the blown-up three sphere. Back in the gauge theory, this corresponds to N mesons receiving the *same* VEVs. The RG flow triggered by the VEV will end in the $\mathcal{N}=4$ SU(N) SYM gauge theory, just because we put the $D3$-branes at a regular point. The backreaction of the brane stack yields the AdS_5 throat, so the entire supergravity solution describes the flow[1] from $AdS_5 \times T^{1,1}$ to $AdS_5 \times S^5$.

Our approach is partially based on the work of Klebanov and Murugan [12]. (See [23, 22, 21] also for closely related work.). They studied a similar emergence of the AdS_5 throat due to a $D3$-brane stack located at a point on the blown-up two sphere of the resolved conifold.[2] On the gauge theory side this describes an RG flow along the *non-mesonic* branch.

Finding the full 10d solution in [12] was equivalent to solving the 6d Laplace equation with a source for the warp function. In our case, we want to add source D3-branes instead to the Klebanov-Strassler background, which is more complicated because of the extra fields etc. But when we add sources, the 6d inhomogeneous Laplace equation is still the only equation we need to consider, because we are still working within the framework of the standard D3-brane ansatz. Indeed, the KS warp function satisfies:

$$\Box_6 h_{\mathrm{KS}} = g_s \star_6 H_3^{\mathrm{KS}} \wedge F_3^{\mathrm{KS}}, \tag{1.1}$$

where H_3^{KS} and F_3^{KS} are the NS-NS and RR 3-forms. There is no source term on the right hand side of the equation, which shows that there are no $D3$-branes in the background, but rather only M fractional branes. On the other hand, the corresponding RR charge $\widetilde{N}$ is non zero and the asymptotic behavior of the self-dual RR-form is:

$$\widetilde{F}_5 \approx \widetilde{N}\,\mathrm{Vol}\left(T^{1,1}\right), \qquad \text{where} \qquad \widetilde{N} = \frac{3}{2\pi} g_s \ln\frac{r}{r_0} \cdot M. \tag{1.2}$$

To build our background we have to split the warp function into two terms:

$$h = h_{\mathrm{KS}} + H_{\mathrm{D3}}, \tag{1.3}$$

where H_{D3} (or simply H throughtout the paper) is the solution of the Laplace equation with the $D3$-brane source:

$$\Box_6 H_{\mathrm{D3}} = N\delta_6(\mathrm{NP}), \tag{1.4}$$

[1]To be more precise the KS solution reproduces the $AdS_5 \times T^{1,1}$ geometry in the UV only up to logarithmic corrections, which just indicates the fact that in the KS model the gauge theory in the UV is not SU(N) $\times$ SU(N).

[2]The singular solution corresponding to $D3$-branes smeared on the S^2 was investigated in [13].

where NP stands for the North pole of the S^3. Now we have:

$$\widetilde{N} = N + \frac{3}{2\pi} g_s \ln \frac{r}{r_0} \cdot M. \tag{1.5}$$

It is essential to notice that the addition of the source term is consistent with the usual ansatz for D3-branes. In particular, the dilaton is constant, the 0-form vanishes and SUSY is not broken.

The organization of the paper is as follows. In the next section and in one of the appendices, we describe the deformed conifold. We introduce a new map which for a given point on the deformed conifold provides its S^3 and S^2 coordinates. The map generalizes the results of [14] for the singular conifold case. We then relate this map to the coordinates introduced in [15] and later used in [16]. These coordinates are different from the standard coordinates used by Klebanov and Strassler [19, 6], and prove to be very convenient for working with the Laplace equation. We explain this in detail in section 3, where we present the numeric solution of the equation and demonstrate the emergence of the AdS throat. Section 4 is devoted to the gauge theory. We show how the cubic superpotential of the $\mathcal{N} = 4$ theory emerges when one expands the mesonic fields around the VEV corresponding to the North pole of the three-sphere. We end with some remarks in section 5. In particular we propose why the gravity mode [8] dual to the Goldstone boson of the baryonic symmetry does not exist in our case. Some of the technicalities have been relegated to various appendices.

2. The deformed conifold

We start with a brief description of the (singular) conifold. In the physics community[3] the word conifold refers to the singular non-compact Calabi-Yau three-fold defined by the complex quadratic equation

$$\sum_{i=1}^{4} z_i^2 = 0. \tag{2.1}$$

This equation represents a real cone over a five-dimensional Einstein manifold called $T^{1,1}$, which is the coset space $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\,\mathrm{U}(1)$. The base $T^{1,1}$ has the topology of $S^2 \times S^3$ [20], and if we denote the metric on it by $d\Omega^2_{T^{1,1}}$ then the full conifold metric takes the standard form of a cone: $ds_6^2 = dr^2 + r^2 d\Omega^2_{T^{1,1}}$.

The singularity at the apex of the conifold can be smoothed in two ways while still respecting the Calabi-Yau condition as explained in the introduction. We will be studying the deformed case here, the resolved conifold has been subjected to a similar study in [12]. For a review of the various conifolds, see the appendices of [17]. A schematic picture of the conifold is in figure 1.

[3] In mathematics, the notion of a conifold is more general. It refers to a generalization of the notion of a manifold, where we allow conical singularities. The physics-conifold is a special case of the mathematics-conifold.

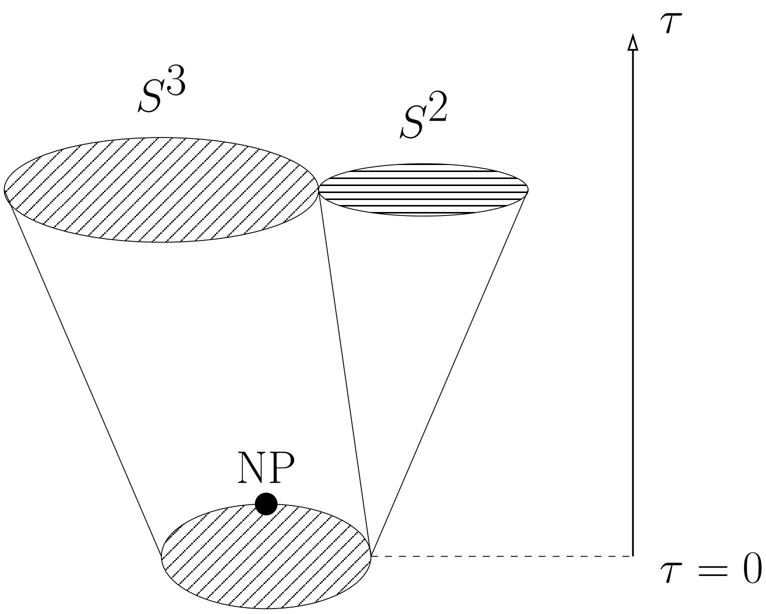

Figure 1: A schematic picture of the (deformed) conifold. NP stands for the North Pole of the non-vanishing three sphere at the tip. Our D-branes are at NP.

The deformation of the conifold is defined by

$$\sum_{i=1}^{4} z_i^2 = \epsilon^2, \tag{2.2}$$

which can be rewritten with an eye for useful future parametrizations as

$$\det W = -\frac{\epsilon^2}{2}, \text{ where } W \equiv \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} z_3 + iz_4 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 + iz_4 \end{pmatrix}. \tag{2.3}$$

By looking at the situation when all z_i are real, it is clear that the S^3 does not vanish at the tip. Also worth noticing is the fact that the deformation breaks the $z_i \to e^{i\alpha} z_i$ symmetry of the singular conifold down to $z_i \to -z_i$. So the deformed conifold does not have the full U(1), but only a $\mathbb{Z}_2$. The radius of the three sphere can be taken as

$$r^2 \equiv \sum_{i=1}^{4} |z_i|^2. \tag{2.4}$$

It should be noted that this r does *not* reduce to the radial coordinate of the cone in the undeformed limit. In fact, if we defined such a radial-like coordinate (i.e., a coordinate that tends to the radial coordinate of the undeformed conifold, far away from the deformation) it would behave as $\tilde{r} \sim r^{2/3}$. We will use this information later.

It is customary to use a new coordinate τ such that $r^2 = \epsilon^2 \cosh\tau$, in terms of which the above equation becomes

$$\mathrm{Tr}(W^\dagger W) = \epsilon^2 \cosh\tau. \tag{2.5}$$

The tip where the S^2 shrinks to zero corresponds to $\tau = 0$.

Part of our purpose in the rest of this paper will be to use the metric to find the explicit supergravity solution that corresponds to a stack of D3-branes localized on the non-vanishing S^3. The D3-branes back-react and warp the geometry and we want to calculate the warp factor. To do this, we will need the Laplacian on the deformed conifold and it will be convenient to have a parametrization of W where the split between the S^2 and the S^3 is explicit. The usual form in which the deformed conifold metric is written down does not have this advantage, so now we consider a system of coordinates where this split is manifest.

The aim is to package the information in the matrix W into two separate pieces which can be interpreted as the S^2 and the S^3. We start with the observation that the hermitian matrix $W^\dagger W$ has two real *positive* eigenvalues:

$$\lambda_1^2 = \frac{\epsilon^2}{2} e^{\tau} \qquad \text{and} \qquad \lambda_2^2 = \frac{\epsilon^2}{2} e^{-\tau}. \tag{2.6}$$

Taking positive square roots of λ_1^2 and λ_2^2 we can define a hermitian non-singular matrix $P \equiv \left(W^\dagger W\right)^{1/2}$ with the eigenvalues $\lambda_1, \lambda_2 > 0$. This matrix, in turn, can be diagonalized:

$$P = U D(\tau) U^\dagger, \qquad \text{where} \qquad D(\tau) \equiv \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.7}$$

and U is an SU(2) matrix. Clearly P is invariant under $U \to e^{i\alpha\sigma_3} U$ for any α, so we have to quotient U by this U(1) action, which is just the Hopf projection from SU(2) ($= S^3$) to S^2. Thus U describes the S^2. To build the S^3 we define a new matrix X:

$$X \equiv -iWP^{-1}. \tag{2.8}$$

It is easy to check that X is unitary and special, so $X \in \mathrm{SU}(2) = S^3$.

To summarize, for fixed τ we built a map from W to U and X which defines the S^2 and the S^3 respectively. The map is invertible and simply given by:

$$W = iXP = iXUD(\tau)U^\dagger. \tag{2.9}$$

Moreover, for $\tau = 0$ we find $P \propto \mathbb{I}_{2\times 2}$ and so U is ill-defined, which, as expected, means that for $\tau = 0$ the two-sphere shrinks to zero size. Furthermore, for $\tau \to \infty$ we have $P = r\,(\mathbb{I}_{2\times 2} + iQ)$, where Q is 2×2 unitary anti-hermitian matrix and therefore the formula (2.9) re-produces the trivialization of the singular conifold proposed in [14].

Now, we wish to write the deformed conifold metric not in terms of the original coordinates which mix the S^2 and S^3, but in terms of the coordinates that manifest the split. This is easily done because we just have to parametrize U in terms of the angles of the two-sphere, and X in terms of the angles of the three-sphere. We will follow the notations

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of [18] and introduce two matrices T and S which are equivalent to our X and U.[4] We have:

$$X = -iT\sigma_3 \qquad \text{and} \qquad U = \sigma_3 S \sigma_3, \quad \text{where} \quad S = e^{\frac{i}{2}\phi\sigma_3} e^{-\frac{i}{2}\theta\sigma_2}. \tag{2.10}$$

This last bit defines a specific angular parametrization on the S^2 in terms of θ and ϕ. Once we also make a parametrization of the SU(2) matrix T in terms of the three angles of S^3 (which we write down in appendix A), we will be done, and have explicit coordinates on the deformed conifold in terms of τ, the three-sphere angles, and the two-sphere angles. Moreover, since it is well-known how to write W (and therefore X and U) in terms of the standard Klebanov-Strassler coordinates, we also have an explicit transformation relating the two coordinate systems.

To write the metric in a convenient form, we use the Maurer-Cartan forms $w_{i=1,2,3}$ on the three-sphere, defined by

$$T^\dagger \mathrm{d}T = \frac{i}{2}\sigma_i w_i. \tag{2.11}$$

In terms of these angle coordinates and using (A.9), the deformed conifold metric takes the following form:

$$\begin{aligned}\epsilon^{-4/3}\mathrm{d}s^2_{(6)} = \frac{1}{6K^2(\tau)}\left(\mathrm{d}\tau^2 + h_3^2\right) + \frac{K(\tau)}{4}\cosh^2\left(\frac{\tau}{2}\right)\Bigg[h_1^2 + h_2^2 + \\ +4\tanh^2\left(\frac{\tau}{2}\right)\left(\left(\mathrm{d}\theta - \frac{1}{2}h_2\right)^2 + \left(\sin\theta\mathrm{d}\phi - \frac{1}{2}h_1\right)^2\right)\Bigg].\end{aligned} \tag{2.12}$$

Here

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh(\tau)} \tag{2.13}$$

and the forms $h_{i=1,2,3}$ are defined by:[5]

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 & \cos\theta & -\sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sin\phi & \cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}. \tag{2.14}$$

The two SO(3) matrices in (2.14) reflect the fact that the three-sphere is fibered over the two-sphere. This fiber is trivial as one can easily verify by properly calculating the Chern class of the fiber bundle [14]. We explicitly write down the h_i in appendix A in terms of the angles of S^3.

From the metric (2.12) it is clear that at $\tau = 0$ the size of the S^2 parameterized by θ and ϕ smoothly shrinks to zero:[6]

$$\epsilon^{-\frac{4}{3}}\mathrm{d}s^2_{(6)} \approx \frac{1}{4}\left(\frac{2}{3}\right)^{\frac{1}{3}}\left[\sum_{i=1}^{3} w_i^2 + \mathrm{d}\tau^2 + \tau^2\left(\left(\mathrm{d}\theta - \frac{h_2}{2}\right)^2 + \left(\sin\theta\mathrm{d}\phi - \frac{h_1}{2}\right)^2\right)\right]. \tag{2.15}$$

[4] Notice that what we have provided essentially is an explicit construction of the S and T matrices of [18] in terms of the standard conifold coordinates, captured by W.

[5] These forms are related to the analogous forms used in [18] as follows: $h_1 = \sqrt{2}\tilde{g}^3$, $h_2 = \sqrt{2}\tilde{g}^4$ and $h_3 = \tilde{g}^5$. Notice also that $\sum_{i=1}^3 h_i^2 = \sum_{i=1}^3 w_i^2$.

[6] Note that $K(\tau = 0) = \left(\frac{2}{3}\right)^{1/3}$.

3. D3-brane supergravity

The type IIB supergravity solution with D3-brane sources is fully specified once we solve the Poisson-type equation for the warp factor on the 6d space. We are interested in putting the stack of branes at the deformed tip, where the S^2 has collapsed to zero size. This means that we can look for the warp factor which is independent of θ and ϕ. On top of that, without loss of generality, we will put the D-branes at the North pole of the S^3, so that only the angle α (see appendix A) will make its appearance in the warp factor. Arguments entirely analogous to this were made in [21] in a different context, where a more detailed discussion can be found. Using the Laplacian written down in appendix B, and the above-mentioned simplifications, the final form of the warp-factor equation that we need to solve is

$$\Box_\tau H + \frac{1}{A^2(\tau)}\frac{1}{\sin^2\alpha}\partial_\alpha(\sin^2\alpha\ \partial_\alpha H) = -\frac{6C}{\pi^2\epsilon^4\sinh^2\tau\sin^2\alpha}\delta(\tau-\tau_0)\delta(\alpha). \tag{3.1}$$

The stack is at $\tau_0 = 0$. $\Box_\tau$ and $A(\tau)$ are defined in appendix B. The general strategy for fixing the normalization of such delta functions and solving equations of this kind can be found in [21]. Here, $C = (2\pi)^4 g_s N\alpha'^2$, N is the number of D3-branes. It is useful also to notice that the determinant of the 6d metric is

$$\sqrt{g_6} = \frac{\epsilon^4}{96}\sinh^2\tau\sin^2\alpha\sin\beta\sin\theta, \tag{3.2}$$

where α and β are the first two angles of the S^3 and θ is the first angle (the latitude) of the S^2 (See appendix A).

We first solve the angle part and look for solutions of

$$\frac{1}{\sin^2\alpha}\partial_\alpha(\sin^2\alpha\ \partial_\alpha Y_l) + l(l+2)Y_l = 0. \tag{3.3}$$

We have chosen this form because energy eigenvalues of the d-sphere are of the form $l(l+d-1)$. The solutions of this three-sphere equation are in fact *simpler* than those of the familiar two-sphere, where the Y_l take the well-known Legendre form. Here instead, we can take the independent solutions in the form

$$Y_l(\alpha) \sim \frac{\cos((l+1)\alpha)}{\sin\alpha},\quad \frac{\sin((l+1)\alpha)}{\sin\alpha}. \tag{3.4}$$

Of the two, since the right hand side of (3.1) is even under $\alpha \leftrightarrow -\alpha$, we will only need the second set to do our expansions. We can fix the normalization by setting

$$\int_0^\pi Y_l(\alpha)Y_{l'}(\alpha)\sin^2\alpha\ d\alpha = \delta_{ll'}. \tag{3.5}$$

The weight comes from the normalization of the delta function in the warp factor equation above. This fixes

$$Y_l(\alpha) = \sqrt{\frac{2}{\pi}}\frac{\sin((l+1)\alpha)}{\sin\alpha}. \tag{3.6}$$

Now, we turn to the radial equation, which takes the formidable shape

$$\Box_\tau H_l - \frac{l(l+2)}{A^2(\tau)}H_l(\tau) = -\frac{6C}{\pi^2\epsilon^4\sinh^2\tau}\delta(\tau-\tau_0). \tag{3.7}$$

We have been able to solve this equation for generic l only numerically.[7] To fully fix a second order differential equation, we need two pieces of data (e.g.: the value of the function at two different points or the value of the function and its derivative at the same point.). The homogeneous equation only determines the solution upto an overall constant, even after one stipulates that it die down at infinity. This overall normalization is fixed by the strength of the delta-function discontinuity at the origin. In particular, in our case it turns out that this gives,

$$\lim_{\tau\to 0}\left[H_l'(\tau)\,(\sinh(2\tau)-2\tau)^{2/3}\right] = -\frac{2^{2/3}}{\pi^2\epsilon^{8/3}}C. \tag{3.9}$$

So one numerical consistency check we can do on our solutions is to check that the left hand side has a good limit as $\tau \to 0$.

We can do another check. We can solve the asymptotic ($\tau \to \infty$) form of the differential equation *exactly*. The asymptotic (homogeneous) equation takes the form

$$h_l''(\tau) + \frac{4}{3}h_l'(\tau) - \frac{4}{3}l(l+2)h_l(\tau) = 0. \tag{3.10}$$

The dying solutions of this equation are

$$h_l(\tau) \sim \exp\left[-\frac{2\tau}{3}\big(1+\sqrt{1+6l+3l^2}\big)\right], \tag{3.11}$$

which implies that

$$\lim_{\tau\to\infty}\frac{H_l'(\tau)}{H_l(\tau)} = -\frac{2}{3}\big(1+\sqrt{1+6l+3l^2}\big). \tag{3.12}$$

This is easily checked numerically, and indeed we have checked that it is satisfied for our solutions. A plot of the radial solutions for some values of l are given in figure 2.

The full solution then, can be written as

$$H(\tau,\alpha) = \sum_{l=0}^{\infty} H_l(\tau)\; Y_l(\alpha)\; Y_l(\alpha_0=0) = \frac{2}{\pi}\sum_{l=0}^{\infty}(l+1)H_l(\tau)\frac{\sin\left((l+1)\alpha\right)}{\sin\alpha}. \tag{3.13}$$

One rather basic consistency check that we can do with this full solution is to compare it to the smeared approximation: the $l=0$ term of the above sum should reproduce the results obtained by assuming that the D3-branes were smeared on the 3-sphere. When the branes are smeared, the warp factor equation is the Laplace equation with all angular dependence suppressed, which reduces to $\Box_\tau H = 0$. The overall normalization can be fixed either by comparison with undeformed conifold in the asymptotic region, or by being careful about the normalization of the delta function source. This normalization essentially just amounts to an extra factor of $\frac{2}{\pi}$ from the integration of the $\sin^2\alpha$ that was there in

[7] For $l = 0$, there is a slight simplification. The solution can be written as

$$H_{l=0}(\tau) \sim \int^{\tau} \frac{1}{(\sinh 2x - 2x)^{2/3}}\mathrm{d}x. \tag{3.8}$$

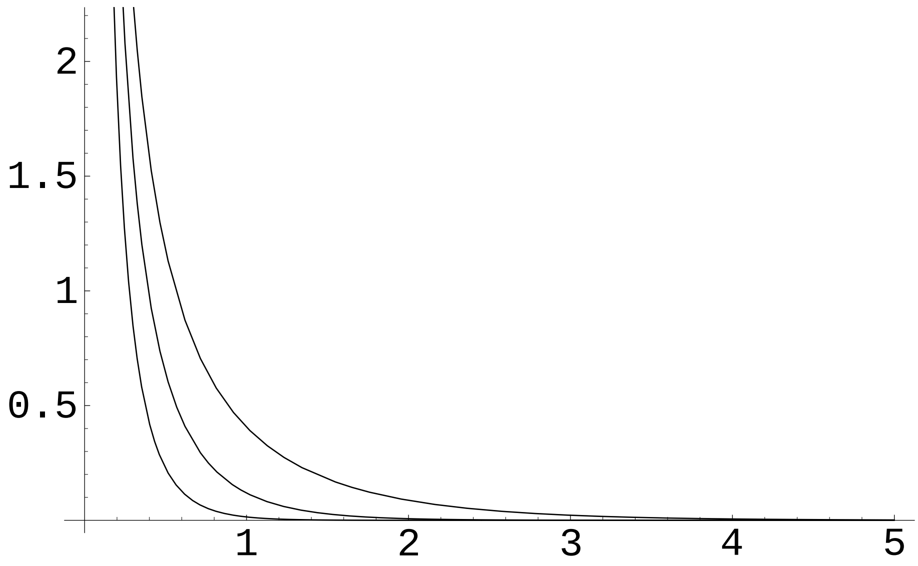

Figure 2: Plots of $\alpha H_l(\tau)$ where $\frac{1}{\alpha} = \frac{2^{2/3}}{\pi^2 \epsilon^{8/3}} C$, for $l = 0, 1, 3$. The curve rises as l increases.

the delta function before the smearing. But from (3.13), we see that it is precisely a factor of $\frac{2}{\pi}$ that multiplies the $H_l(\tau)$, when $l = 0$.

Using the warp factor, one can also demonstrate the emergence of the AdS throat close to the D-brane stack. This is easiest to do along $\alpha = 0$.[8] There the warp factor (3.13) takes the form $\sim \sum (l+1)^2 H_l(\tau)$. Now, we can take a "near-horizon" limit of (3.7) and solve it exactly to find what $H_l(\tau)$ looks like close to the stack. It turns out that the (homogeneous) near-horizon radial equation is

$$H''(\tau) + \frac{2H'(\tau)}{\tau} - l(l+2)H(\tau) = 0, \text{ with solution } H(\tau) \sim \frac{e^{-\sqrt{l(l+2)}\tau}}{\tau}. \tag{3.14}$$

It turns out that the normalization of the solution (fixed by integrating across the source) is independent of l, so the *entire* dependence on l and τ is captured by the above expression, which we write schematically as $\frac{f(l\tau)}{\tau}$. Since the sum over all l's must converge, we can think of this as a regulator [12] and write

$$H(r) \sim \sum_{l=0}^{\infty} l^2 \frac{f(l\tau)}{\tau} \sim \sum_{l=0}^{1/\tau} l^2 \frac{f(l\tau)}{\tau} \sim \int_0^{1/\tau} l^2 \frac{f(l\tau)}{\tau} dl \sim \frac{\int_0^1 x^2 f(x) dx}{\tau^4} \sim \frac{\text{const.}}{\tau^4}. \tag{3.15}$$

Since the radial coordinate looks like τ near $\tau \sim 0$ (see footnote), this means that in the near horizon region, in terms of the flat coordinate, the warp factor goes as $\sim \frac{1}{\tau^4}$. But this is of course what gives rise to the origin of the AdS throat.

[8]We can expand the deformed conifold metric from the previous section when $\tau, \alpha \ll 1$. The radial coordinate turns out to be of the form $\sim \sqrt{\tau^2 + \alpha^2}$ upto irrelevant numerical factors. By restricting to a flow along which $\alpha = 0$, our radial coordinate takes the simpler form τ.

4. The dual gauge theory and the mesonic branch

As we have explained in the Introduction our supergravity solution describes a stuck of N $D3$-branes located at the "tip" of the deformed conifold. The gauge theory dual to this solution was analyzed both in the original paper [6] and in more detail in [10]. The dual theory has an $\mathrm{SU}(\widetilde{N}+M)\times\mathrm{SU}(\widetilde{N})$ gauge group, where $\widetilde{N}$ is related to M and N as in (1.5). As the $\mathrm{SU}(\widetilde{N}+M)$ gauge group becomes strongly coupled in the IR, it is described effectively by four mesons $M_{\alpha\beta}=A_\alpha B_\beta$, where $A_{1,2}$ and $B_{1,2}$ are the bi-fundamental chiral fields. The theory is Seiberg dual to a theory with an $\mathrm{SU}(\widetilde{N})\times\mathrm{SU}(\widetilde{N}-M)$ gauge group, where now the first factor becomes strongly coupled in the IR, and the field content is given by the dual "magnetic" quarks, which play now the role of the bi-fundamental fields. For each step of the cascade, therefore, we have $\widetilde{N}\rightarrow\widetilde{N}-M$. In general, the duality cascade proceeds until $\widetilde{N}$ becomes smaller than M, where the dual description does not exist and instead the quantum corrections are captured by the non-perturbative Affleck-Dine-Seiberg (ADS) term in the superpotential. We are, however, interested in a case where the cascade stops at $\widetilde{N}=N$, due to the mesons acquiring large enough VEVs. In this situation the classical superpotential also receives a non-perturbative ADS like contribution:

$$W=h\mathrm{Tr}\left(\mathcal{M}_{11}\mathcal{M}_{22}-\mathcal{M}_{12}\mathcal{M}_{21}\right)+(M-N)\left(\frac{\Lambda^{N+3M}}{\det_{ab\alpha\beta}\mathcal{M}}\right)^{\frac{1}{M-N}}. \tag{4.1}$$

Here the first term is the classical superpotential. Notice that for $N>M$ the determinant appears actually with a positive power. It becomes a real ADS potential only for $N<M$. In any case, however, the moduli space describes N $D3$-branes moving on the deformed conifold. Indeed, the equations of motion for the mesons imply that all the matrices $M_{\alpha\beta}$ commute and also:

$$\begin{aligned}
\det_{ab\alpha\beta}\mathcal{M} &= \left(h^{\left(\frac{N}{M}-1\right)}\Lambda^{\left(\frac{N}{M}+3\right)}\right)^N \\
\mathrm{Tr}_{ab}\det_{\alpha\beta}\mathcal{M} &= Nh^{\left(\frac{N}{M}-1\right)}\Lambda^{\left(\frac{N}{M}+3\right)}.
\end{aligned} \tag{4.2}$$

The matrices $M_{\alpha\beta}$ can be all simultaneousely diagonalized. The above equations will then both lead to the deformed conifold definition for the eigenvalues of $M_{\alpha\beta}$'s with the deformation parameter ϵ being a function of h and Λ:

$$\epsilon^2\propto h^{\left(\frac{N}{M}-1\right)}\Lambda^{\left(\frac{N}{M}+3\right)}. \tag{4.3}$$

Computations similar to what we have done above can be found, for example, in [24, 25]. In the latter, matrix model techniques were applied for steups with more than one conifold singularity.

In this paper we have constructed a supergravity solution dual to the mesonic branch of the gauge theory. The $D3$-brane source in our picture is located at $\tau=0$ (the minimal value of the radial cootdinate), where the two sphere smoothly shrinks to zero. For the $10d$ solution to be regular the $D3$-branes have to be localized at a point, which in our conventions is the North pole of the non-shrinking three-sphere. On the gauge theory side

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it means that all the eigenvalues $m^i_{\alpha\beta}$ of the matrix $\mathcal{M}_{\alpha\beta}$ are the same and correspond to the North pole of the S^3 as we have explained in section 2. An RG flow triggered by the VEVs leads in the IR to the $\mathcal{N}=4$ SYM theory, which on the supergravity side corresponds to AdS throat developed near the $D3$-brane source. In the rest of this section we want to show that expanding the superpotential (4.1) around the VEV corresponding to the North pole we find, as expected, the cubic superpotential of the $\mathcal{N}=4$ SYM.

In the coordinates introduced in appendix A the North pole corresponds to $\alpha=0$ and thus $X=\sigma_0$. On the other hand $D(\tau=0)=\frac{\epsilon}{\sqrt{2}}\sigma_0$ and so (2.9) implies that $W=i\frac{\epsilon}{\sqrt{2}}\sigma_0$. We know that $m^i_{\alpha\beta}$'s are related to $w_{\alpha\beta}$'s so we have to consider the following VEVs:

$$\langle\mathcal{M}_{11}\rangle = i\frac{\epsilon}{\sqrt{2}}\cdot\mathbb{I}_{N\times N}, \quad \langle\mathcal{M}_{22}\rangle = i\frac{\epsilon}{\sqrt{2}}\cdot\mathbb{I}_{N\times N}, \quad \langle\mathcal{M}_{12}\rangle = 0, \quad \text{and} \quad \langle\mathcal{M}_{21}\rangle = 0. \tag{4.4}$$

Next we will consider the expansion around the VEV:

$$\begin{aligned} \mathcal{M}_{11} &= \langle\mathcal{M}_{11}\rangle + \delta\cdot(\Phi-\Phi_1)\,, & \mathcal{M}_{22} &= \langle\mathcal{M}_{22}\rangle + \delta\cdot(\Phi+\Phi_1)\,,\\ \mathcal{M}_{12} &= \langle\mathcal{M}_{12}\rangle + \delta\cdot\Phi_2, & \mathcal{M}_{12} &= \langle\mathcal{M}_{21}\rangle + \delta\cdot\Phi_3, \end{aligned} \tag{4.5}$$

where

$$\delta^2 \equiv h^{\left(-1+\frac{N}{3M}\right)}\Lambda^{\left(1+\frac{N}{3M}\right)}. \tag{4.6}$$

Up to the quartic terms this yields:

$$W = \text{const} + \text{Tr}\left(\Phi_1\left[\Phi_2,\Phi_3\right]\right) + 2h\delta^2\text{Tr}\Phi^2 + \frac{2}{3}\text{Tr}\Phi^3 + \text{Tr}\left(\Phi\left\{\Phi_2,\Phi_3\right\}\right) + 2\text{Tr}\left(\Phi\Phi_1^2\right)\ldots \tag{4.7}$$

Here we made use of the formulae collected in appendix C. The first non-trivial term here is exactly the $\mathcal{N}=4$ SYM cubic superpotential. Notice that the fields Φ_i have dimension one as it should be in the conformal $\mathcal{N}=4$ theory. It follows from the fact that $\mathcal{M}_{\alpha\beta}$ have dimension two and the parameter δ has dimension one. The remaining field Φ is massive. This is expected, since the deformed conifold is a three-dimensional embedding in $\mathbb{C}^4$ and Φ describes the only direction, which is not tangent to the conifold. Thus this field is also not tangent to the moduli space and is expected to be massive. One can easily check that integrating out Φ produces quartic Φ_i terms, which, of course, become irrelevant in the IR.

5. Concluding remarks

In this paper we have constructed a supergravity background dual to the mesonic branch of the gauge theory. We therefore do not expect the $\text{U}(1)_B$ to be broken. The baryonic symmetry is not related to one of the background isometries, it rather appears as a gauge symmetry of the Wess-Zumino term. Still one can ask whether the Goldstone boson mode found in [8] ceases to exist once we add the $D3$-brane source. If the mode does not exist anymore we can safely assume that the baryonic symmetry is unbroken. This indeed seems to be the case, since most of the expressions (for example equation (3.25)) in [8] explicitly include the warp function h. For $h=h_{\text{KS}}$ these expressions are normalizable at $\tau\to 0$ with respect to the conifold metric. However, for $h=h_{\text{KS}}+H_{\text{D3}}$ most of these expressions will diverge, since near the North pole at $\tau=0$ we have $h\approx H_{\text{D3}}\approx\frac{1}{\tau^4}$. It will be interesting to make this statement more rigorous proving therefore that our background is indeed related to the mesonic branch of the gauge theory, where no Goldstone boson is expected.

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A. Technicalities: S^2, S^3 and the Kähler metric

We complete the definition of S^2 and S^3 here by giving the explicit matrices. Using the matrix S defined in (2.10) the matrix P becomes:

$$P = \frac{\epsilon}{\sqrt{2}} \begin{pmatrix} e^{\frac{\tau}{2}} \cos^2\frac{\theta}{2} + \ e^{-\frac{\tau}{2}} \sin^2\frac{\theta}{2} & \sinh(\frac{\tau}{2}) e^{i\phi} \sin\theta \\ \sinh(\frac{\tau}{2}) e^{-i\phi} \sin\theta & e^{-\frac{\tau}{2}} \cos^2\frac{\theta}{2} + \ e^{\frac{\tau}{2}} \sin^2\frac{\theta}{2} \end{pmatrix}. \tag{A.1}$$

As for the three-sphere S^3, it is defined by the real numbers satisfying

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1. \tag{A.2}$$

This is identical to the group SU(2) because (A.2) is precisely the condition that turns a general 2×2 matrix X defined by

$$X = \begin{pmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{pmatrix} \tag{A.3}$$

into a special, unitary 2×2 matrix. Using the fact that SU(2) is a group, we can use its Maurer-Cartan one form

$$X^\dagger dX \equiv \frac{i}{2} \sum_{i=1}^{3} w_i \sigma_i, \tag{A.4}$$

to define a basis of canonical one-forms on S^3. If we parametrize S^3 in the usual way

$$x_0 = \cos\alpha,\ x_1 = \sin\alpha\cos\beta,\ x_2 = \sin\alpha\sin\beta\cos\gamma,\ x_3 = \sin\alpha\sin\beta\sin\gamma, \tag{A.5}$$

then, by explicit computation using the above formulae, we find

$$\frac{w_1}{2} = \cos\beta d\alpha - \sin\alpha\cos\alpha\sin\beta d\beta + \sin^2\alpha\sin^2\beta d\gamma, \tag{A.6}$$

$$\begin{aligned}\frac{w_2}{2} &= \sin\beta\cos\gamma d\alpha + (\sin\alpha\cos\alpha\cos\beta\cos\gamma - \sin^2\alpha\sin\gamma) d\beta + \\ &\qquad -(\sin^2\alpha\sin\beta\cos\beta\cos\gamma + \sin\alpha\cos\alpha\sin\beta\sin\gamma) d\gamma,\end{aligned} \tag{A.7}$$

$$\begin{aligned}\frac{w_3}{2} &= \sin\beta\sin\gamma d\alpha + (\sin\alpha\cos\alpha\cos\beta\sin\gamma + \sin^2\alpha\cos\gamma) d\beta + \\ &\qquad +(\sin\alpha\cos\alpha\sin\beta\cos\gamma - \sin^2\alpha\sin\beta\cos\beta\sin\gamma) d\gamma.\end{aligned} \tag{A.8}$$

Now we turn to the conventional definition of the metric on the deformed conifold in terms of its Kähler potential. We use this in the derivation of (2.12). The metric can be written in the form [20]:

$$ds_6^2 = \mathcal{F}' \ \mathrm{Tr}(\mathrm{d}W^\dagger \mathrm{d}W) + \mathcal{F}'' \ |\mathrm{Tr}(W^\dagger \mathrm{d}W)|^2, \tag{A.9}$$

where $\mathcal{F} \equiv \mathcal{F}(r^2)$ and

$$\mathcal{F}' \equiv \frac{\partial \mathcal{F}}{\partial r^2} = \frac{1}{\epsilon^2} \frac{1}{\sinh \tau} \frac{\partial \mathcal{F}}{\partial \tau}, \quad \text{with} \quad \frac{\partial \mathcal{F}}{\partial \tau} = 2^{-1/3} \epsilon^{4/3} (\sinh(2\tau) - 2\tau)^{1/3}. \tag{A.10}$$

B. Laplacian in two different coordinates

This appendix is dedicated to writing down the Laplacian for the deformed conifold in the standard coordinates and also in the coordinates where the $S^2 - S^3$ split is manifest. We will need only the latter form, but we present both of them here for the convenience of posterity. In what follows, the functions $A(\tau)$ and $B(\tau)$ are defined by

$$A^2(\tau) = \frac{2^{-1/3}}{8} \coth \frac{\tau}{2} \ (\sinh 2\tau - 2\tau)^{1/3}, \quad B^2(\tau) = \frac{2^{2/3}}{6} \frac{\sinh^2 \tau}{(\sinh 2\tau - 2\tau)^{2/3}}. \tag{B.1}$$

Klebanov-Strassler coordinates: the scalar Laplacian in Klebanov-Strassler coordinates can be written in the form (the notations can be found in [6]):

$$\Box H = \Box_\tau H + f_R(\tau) \Box_R H + f_S(\tau) \big(\Box_1 H + \Box_2 H \big) + f_m(\tau) \Box_m H, \tag{B.2}$$

where

$$f_R(\tau) = \frac{1}{B^2(\tau)}, \quad f_S(\tau) = \frac{\coth^2 \tau}{A^2(\tau)}, \quad f_m(\tau) = \frac{\cosh \tau}{A^2(\tau) \sinh^2 \tau}, \tag{B.3}$$

and

$$\Box_\tau = \frac{\coth^2 \tau}{A^4(\tau) B^2(\tau)} \frac{\partial}{\partial \tau} \left(A^4(\tau) \tanh^2 \tau \frac{\partial}{\partial \tau} \right), \quad \Box_R = \partial_\psi^2, \tag{B.4}$$

$$\Box_i = \frac{1}{\sin \theta_i} \partial_{\theta_i} \ (\sin \theta_i \ \partial_{\theta_i} \ \ ) + \left(\frac{1}{\sin \theta_i} \partial_{\phi_i} - \cot \theta_i \partial_\psi \right)^2. \tag{B.5}$$

The $\Box_i$ arise from the two S^3's (or equivalently, SU(2)'s) that are part of the original $T^{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$. The modding by the U(1) is reflected in the fact that the two S^3 Laplacians share a common angle, ψ. The ugly final piece in (B.2) that could mix the various angular eigenvalues is:

$$\begin{aligned} \frac{1}{2} \Box_m = & -\cos \psi \left(\partial_{\theta_1} \partial_{\theta_2} - \left(\cot \theta_1 \partial_\psi - \frac{\partial_{\phi_1}}{\sin \theta_1} \right) \left(\cot \theta_2 \partial_\psi - \frac{\partial_{\phi_2}}{\sin \theta_2} \right) \right) + \\ & + \sin \psi \left(\left(\cot \theta_2 \partial_\psi - \frac{1}{\sin \theta_2} \partial_{\phi_2} \right) \partial_{\theta_1} + \left(\cot \theta_1 \partial_\psi - \frac{1}{\sin \theta_1} \partial_{\phi_1} \right) \partial_{\theta_2} \right). \end{aligned} \tag{B.6}$$

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$S^2 - S^3$ coordinates: with $\Box_\tau$, $A(\tau)$ and $B(\tau)$ defined as in the previous case, we have:

$$\begin{aligned}\Box H &= \Box_\tau H + \frac{1}{A^2(\tau)}(\partial_1^2+\partial_2^2+\partial_3^2)H + \frac{1+\coth^2\frac{\tau}{2}}{4A^2(\tau)}\left(\frac{\partial_\theta(\sin\theta\partial_\theta H)}{\sin\theta}+\frac{\partial_\phi^2 H}{\sin^2\theta}\right)+\\ &\quad+\frac{1}{A^2(\tau)}\left(\big(\sin\phi\partial_1+\cos\phi\partial_2\big)\partial_\theta H+\big(\cos\theta(\cos\phi\partial_1-\sin\phi\partial_2)-\sin\phi\partial_3\big)\frac{\partial_\phi H}{\sin\theta}\right).\end{aligned} \tag{B.7}$$

Here $\partial_i \equiv \partial_{w_i}$, $i=1,2,3.$. In particular, $(\partial_1^2+\partial_2^2+\partial_3^2)$ is nothing but the S^3 Laplacian. When we put the stack of D3-branes on the non-vanishing S^3, the S^2 has shrunk to zero size and so we can drop terms that have derivatives of H with respect to θ and ϕ.

C. Matrix technology

Let us use the notation $|\mathcal{M}| = \det\mathcal{M}$, where $\mathcal{M}$ is an arbitrary *invertible* square matrix. Then:

$$\begin{aligned}\frac{\delta|\mathcal{M}|^n}{\delta\mathcal{M}_{mn}}\delta\mathcal{M}_{mn} &= n|\mathcal{M}|^n\mathrm{Tr}(\mathcal{M}^{-1}\delta\mathcal{M})\\ \frac{\delta^2|\mathcal{M}|^n}{\delta\mathcal{M}_{mn}\delta\mathcal{M}_{m'n'}}\delta\mathcal{M}_{mn}\mathcal{M}_{m'n'} &= n|\mathcal{M}|^n\left(n\left(\mathrm{Tr}(\mathcal{M}^{-1}\delta\mathcal{M})\right)^2-\mathrm{Tr}\left(\mathcal{M}^{-1}\delta\mathcal{M}\right)^2\right)\end{aligned} \tag{C.1}$$

and

$$\begin{aligned}\frac{\delta^3|\mathcal{M}|^n}{\delta\mathcal{M}_{mn}\delta\mathcal{M}_{m'n'}\delta\mathcal{M}_{m''n''}}\delta\mathcal{M}_{mn}\mathcal{M}_{m'n'}\delta\mathcal{M}_{m''n''} &= n|\mathcal{M}|^n\Big(n^2\left(\mathrm{Tr}(\mathcal{M}^{-1}\delta\mathcal{M})\right)^3-\\ &\quad-3n\left(\mathrm{Tr}(\mathcal{M}^{-1}\delta\mathcal{M})\right)\mathrm{Tr}\left(\mathcal{M}^{-1}\delta\mathcal{M}\right)^2\\ &\quad+2\mathrm{Tr}\left(\mathcal{M}^{-1}\delta\mathcal{M}\right)^3\Big).\end{aligned} \tag{C.2}$$

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